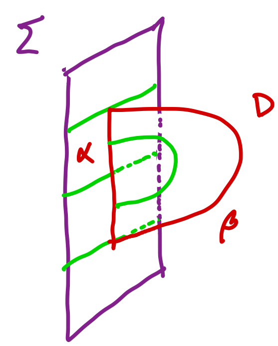
C. Bypasses

let Σ be a convex surface

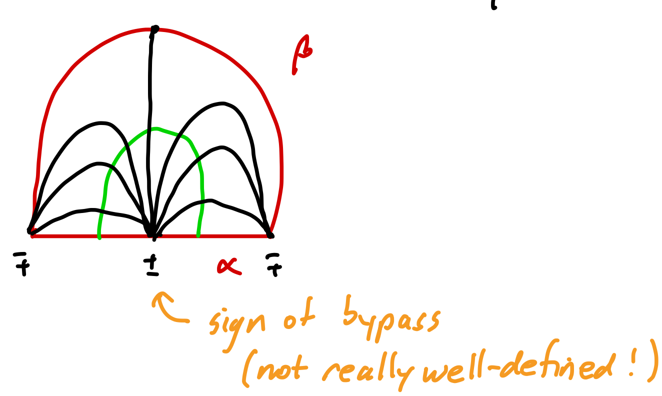
α a Legendrian arc in Σ that intersects the dividing set Γ_Σ (transversely) in 3 points p_1, p_2, p_3 with p_1, p_3 end points

a bypass for Σ (along α) is a disk D s.t.

1) D convex with Legendrian boundary
2) $D \cap \Sigma = \alpha$ (and intersection transverse)
3) $tb(D) = -1$
4) $\partial D = \alpha \cup \beta$
5) $\alpha \cap \beta = p_1 \cup p_3$ are corners of ∂D and elliptic singularities of D

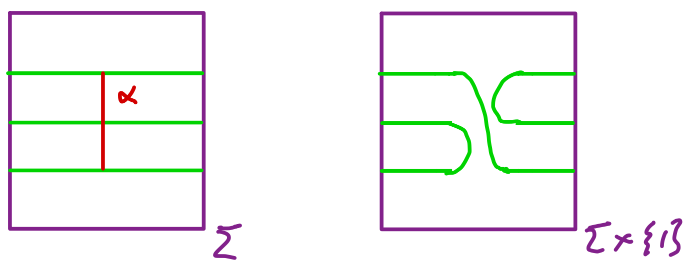


by Giroux flexibility can assume D_ξ is



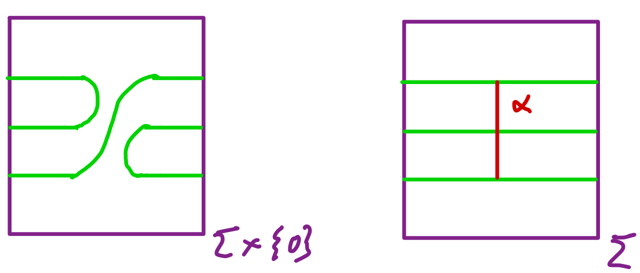
Thm 5:

Σ convex surface
D a bypass for Σ along α
then $\Sigma \cup D$ has a nbhd $N = \Sigma \times [0,1]$
s.t. $\Sigma = \Sigma \times \{0\}$ and
Γ_Σ is related to $\Gamma_{\Sigma \times \{1\}}$ by



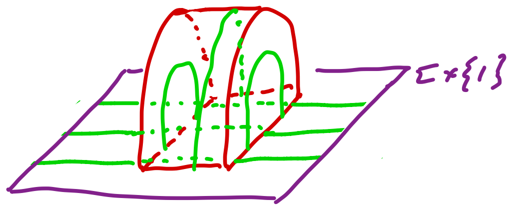
We say $\Sigma \times \{1\}$ is obtained from Σ by <u>attaching a bypass along α from the front</u>

Remark: if $N = \Sigma \times [0,1]$ with $\Sigma = \Sigma \times \{1\}$, then we have

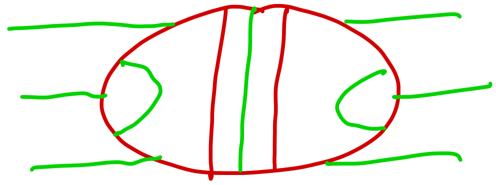


and we say $\Sigma \times \{0\}$ is obtained from Σ by <u>attaching a bypass along Σ from the back</u>

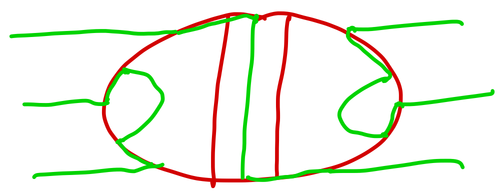
Proof: basic idea is to look at I-invariant nbhd of Σ and D then round corners



flatten picture



round



exercise: carefully work this out

exercise: let Σ' be obtained from Σ by a bypass attachment

show $\quad \chi(\Sigma'_+) - \chi(\Sigma'_-) = \chi(\Sigma_+) - \chi(\Sigma_-)$

Thm 6:

let Σ be a closed convex surface

Σ' be obtained from Σ by a bypass attachment

in a tight contact manifold

Then

(1) $\Gamma_{\Sigma'} = \Gamma_\Sigma$ "trivial" bypass

(2) $\#\Gamma_{\Sigma'} = \#\Gamma_\Sigma + 2$

(3) $\#\Gamma_\Sigma = \#\Gamma_{\Sigma'} - 2$

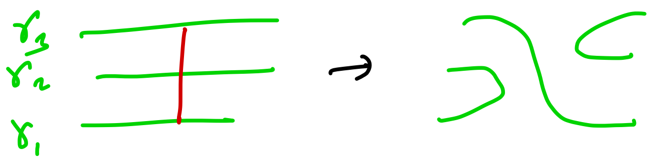
(4) $\Gamma_{\Sigma'}$ is obtained from Γ_Σ by a Dehn twist about some curve in Σ

(5) $\Gamma_{\Sigma'}$ is obtained from Γ_Σ by a "mystrey move" (see figure below)

Proof: consider the points $\{p_1, p_2, p_3\} = \alpha \cap \Gamma_\Sigma$

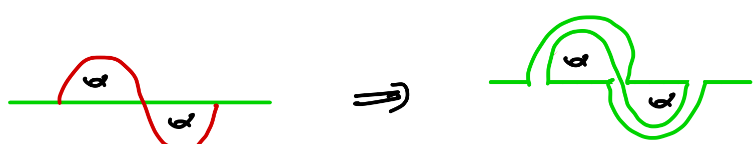
let γ_i be the dividing curve containing p_i

if none of γ_i are same then



all γ_i connected now so case 3)

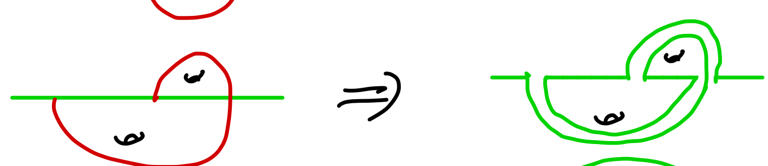
if all γ_i same then we have



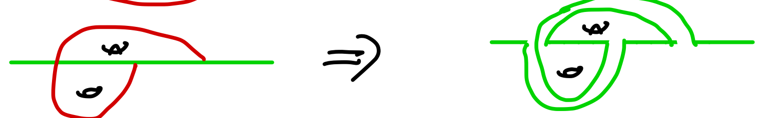
no topology 1)
topology case 4) exercise

case 2)

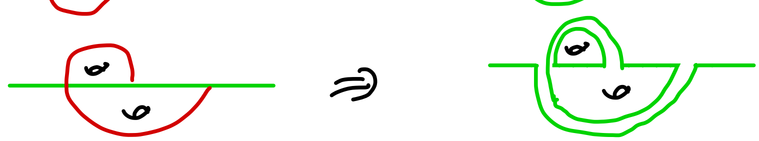
case 2)



no topology case 1)
topology case 4) exercise



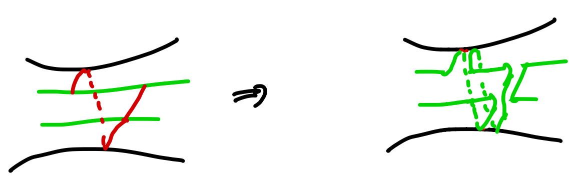
no topology case 1)
topology case 4) exercise



case 2)

if $\gamma_1 = \gamma_3 \neq \gamma_2$

case 4)



case 5)

if $\gamma_1 \neq \gamma_3$ but $\gamma_2 = \gamma_1$ or γ_2

exercise: show you get 1), 4) or 5)

example: S^2 in a tight contact manifold

all bypass attachments are trivial! since Γ_{S^2} must be connected

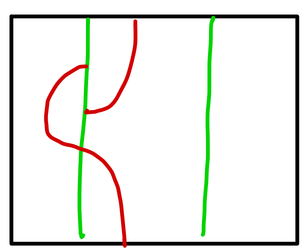
example: T^2 in a tight contact manifold

bypasses can

1) be trivial
2) increase $|\Gamma_{T^2}|$ by 2
3) decrease " "
4) perform a right handed Dehn twist (must have $|\Gamma_\Sigma| = 2$ in this case)

note: in case 1), 2) the attaching arc for the bypass has consecutive intersections with a single dividing curve

eg.



when this does not happen we say the bypass intersects the dividing curves efficiently

Thm 7:

let T be a convex torus in a tight contact manifold
assume T_3 is standard with dividing slope ∞ and ruling slope $r \in [-1, 0)$
and there is a bypass D attached to front of T along a ruling curve
the result of attaching D to T is a convex torus T' s.t.

(1) if $|\Gamma_T|>2$, then $|\Gamma_{T'}|=|\Gamma_T|-2$

(2) if $|\Gamma_T|=2$, then $|\Gamma_{T'}|=2$ and $\text{slope}(\Gamma_{T'})=-1$

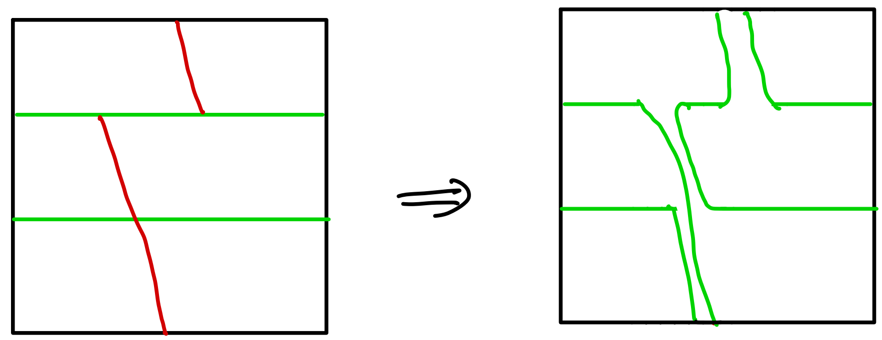
moreover, in case (2) region between T and T' is a basic slice

Proof:

(1) is clear from above

(2)

assume $r\in(-1,0)$



Region R between T and T' satisfies all properties of a basic slice exept need to see minimally twisting

we show this by embedding R is a minimally twisting contact structure

let $(T^2\times[0,1],\xi)$ be the basic slice constructed in the proof of Lemma 3, so

$$\text{slope }\Gamma_{T\times\{0\}}=\infty$$
$$\text{slope }\Gamma_{T\times\{1\}}=-1$$

can arrange $T\times\{0,1\}$ are standard with ruling slope $r=-p/q\in(-1,0)$

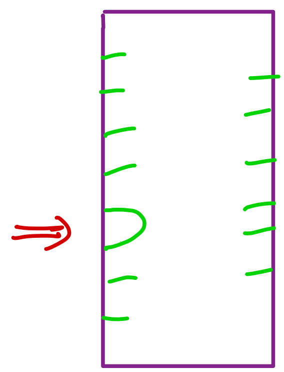
so curve is $q\lambda+p\mu$

note: $|(q\lambda-p\mu)\cdot\mu|=q$ so ruling curve on $T\times\{0\}$ intersects dividing curves $2q$ times

$|(q\lambda - p\mu)\cdot(\lambda-\mu)| = |p-q| < q$ since $-p/q \in (-1,0)$

thus if A is an annulus in $T^2\times[0,1]$ with ∂A a ruling curve on $T^2\times\{0\}$ and one on $T^2\times\{1\}$ then we can make it convex and Γ_A intersect $T^2\times\{0\}$ more than $T^2\times\{1\}$

$\therefore$ must see



we can use Giroux flexibility to realize a bypass B on A for $T^2\times\{0\}$

now $(T^2\times\{0\})\cup B$ has a nbhd contactomorphic to R

$\therefore R$ is minimally twisting and hence a basic slice

exercise: check this carefully might need to consider $-\xi$ if sign of bypass not right

exercise: check $r=-1$ case

Corollary 8:

let T be a convex torus with 2 dividing curves of slope s and ruling slope $r\neq s$

suppose a bypass is attached to the front of T along a ruling curve

let T' be the resulting convex torus

T' will have dividing slope s' where $s'\in[s,r]$ is closest point to r with edge to s

if bypass attached to back of T then slope of T' is $s' \in [r, s]$ where s' closest point to r with an edge to s

Proof:

by choosing the right basis we can assume $s = \infty$

the $r \in [-1, 0)$ case is exactly Thm 7

now suppose $r \in [n, n+1)$ for some $n \in \mathbb{Z}$

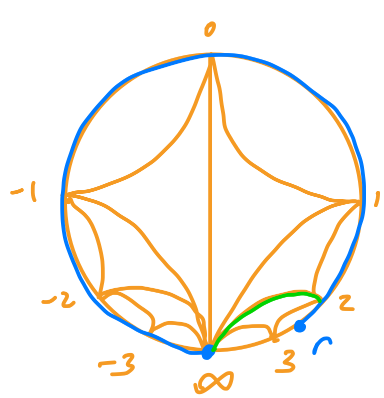
there is a change of basis that fixes ∞ and sends n to -1 (of course $n+1$ goes to 0)

in this basis $r \in [-1, 0)$

attaching a bypass will result in a convex torus of slope -1

but in old basis this will be n

note: n is the point in $[\infty, r)$ closest to r with an edge to ∞



D. Continued fractions and the Farey graph

recall $[a_0; a_1, \dots, a_n]$ denotes

$$r = a_0 - \cfrac{1}{a_1 - \cfrac{\ddots}{\ddots \, - \cfrac{1}{a_n}}}$$

and here we have $a_i \le -2$ for $i > 0$

set $r^a = [a_0; a_1, \dots, a_{n-1}]$ anticlockwise of r

$r^c = [a_0; a_1, \dots, a_n + 1]$ clockwise of r

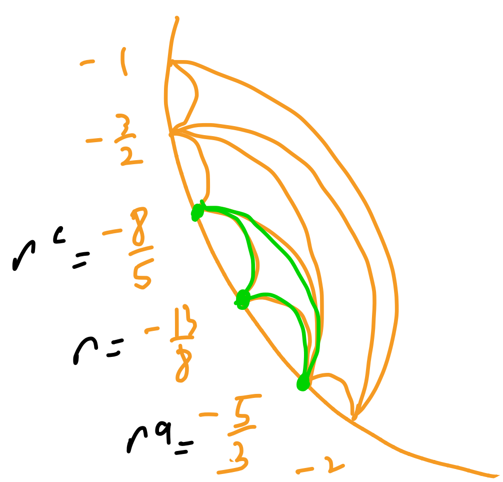
where $[a_0; a_1, \dots, a_k, -1] = [a_0; a_1, \dots, a_k + 1]$

example:

$-\frac{13}{8} = [-2, -3, -3]$

so $\left(-\frac{13}{8}\right)^a = [-2, -3] = \frac{-5}{3}$

$\left(-\frac{13}{8}\right)^c = [-2, -3, -2] = \frac{-8}{5}$



exercise: compute continued fractions of r and compute r^a, r^c for $r = -11/5, -11/9, -7/2, 13/49$

lemma 9

suppose r is <u>not</u> a non-negative integer

the number r^c is the largest rational number bigger than r with an edge to r

the number r^a is the smallest rational number less than r with an edge to r

there is an edge between r^a and r^c and if $r^a = \frac{p^a}{q^a}$ and $r^c = \frac{p^c}{q^c}$ then

$$r = \frac{p^a + p^c}{q^a + q^c}$$

if r is a positive integer then $r^c = \infty$, $r^a = r-1$

before proving lemma we give a few exercises

exercise:

suppose $\frac{p}{q}, \frac{r}{s} \neq 0, \infty$

Show $\frac{p}{q} \cdot \frac{r}{s} = qr - ps = -1 \iff \frac{p}{q}$ and $\frac{r}{s}$ connected by an edge and $\frac{p}{q}$ is clockwise of $\frac{r}{s}$

here we mean if you look at shortest arc $\frac{p}{q}, \frac{r}{s}$ break ∂D^2 into, then along this arc $\frac{p}{q}$ clockwise of $\frac{r}{s}$

and $\frac{p}{q} \cdot \frac{r}{s} = 1 \iff \frac{p}{q}, \frac{r}{s}$ connected by an edge and $\frac{p}{q}$ is anticlockwise of $\frac{r}{s}$

exercise: given $r = [a_0; a_1, \ldots, a_n]$

let $\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k]$ for $k = 0, \ldots, n$

and $p_{-1}=1$, $p_{-2}=0$, $q_{-1}=0$ and $q_{-2}=1$

Show $p_{k+1} = a_{k+1} p_k - p_{k-1}$

$q_{k+1} = a_{k+1} q_k - q_{k-1}$

exercise:

if $\frac{p}{q}$, $\frac{p'}{q'}$ satisfy $\frac{p}{q} \cdot \frac{p'}{q'} = p'q - pq' = \pm 1$ then

$$r - \frac{1}{p/q} = \frac{pr-q}{p} \text{ and } r - \frac{1}{p'/q'} = \frac{p'r-q'}{p'}$$

satisfy $\frac{pr-q}{p} \cdot \frac{p'r-q'}{p'} = \pm 1$

Proof of lemma 9: assume $r = \frac{p_n}{q_n} = [a_0, \dots, a_n]$ is negative

we have $\frac{p_0}{q_0} = \frac{a_0}{1}$

$$\frac{p_1}{q_1} = a_0 - \frac{1}{a_1} = \frac{a_0 a_1 - 1}{a_1}$$

so $\frac{p_0}{q_0} \cdot \frac{p_1}{q_1} = p_1 q_0 - p_0 q_1 = (a_0 a_1 - 1)\cdot 1 - a_0 a_1 = -1$

the exercise above gives

$$\frac{p_{k+1}}{q_{k+1}} \cdot \frac{p_k}{q_k} = p_k q_{k+1} - p_{k+1} q_k$$

$$= p_k(a_{k+1} q_k - q_{k-1}) - q_k(a_{k+1} p_k - p_{k-1})$$

$$= -p_k q_{k-1} + q_k p_{k-1} = \frac{p_k}{q_k} \cdot \frac{p_{k-1}}{q_{k-1}}$$

$\therefore \frac{p_{k+1}}{q_{k+1}} \cdot \frac{p_k}{q_k} = -1$ for all k

and exercise above says there is an edge from

$\frac{p_{k+1}}{q_{k+1}}$ to $\frac{p_k}{q_k}$ and $\frac{p_k}{q_k}$ anticlockwise of $\frac{p_{k+1}}{q_{k+1}}$

i.e. $\frac{p^a}{q^a} = \frac{p_{n-1}}{q_{n-1}}$ is anticlockwise of $\frac{p}{q}$

note: $\frac{a_n}{1} \cdot \frac{a_n+1}{1} = 1$

so exercise above says $[a_{n-1}, a_n] \cdot [a_{n-1}, a_n+1] = 1$

and inductively if $\frac{p^c}{q^c} = [a_0, \dots, a_{n-1}, a_n+1]$

then $\frac{p}{q} \cdot \frac{p^c}{q^c} = 1$

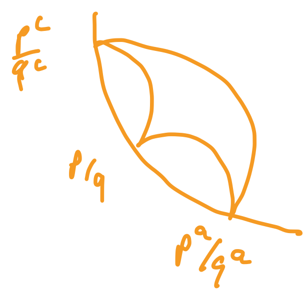
so from above $\frac{p^c}{q^c}$ clockwise of $\frac{p}{q}$ and has an edge to it

finally $\frac{p^a}{q^a} = [a_0, \dots, a_{n-1}]$ is obtained from $\frac{p^c}{q^c} = [a_0, \dots, a_n+1]$

by dropping last entry (if $a_n \neq -2$)

so from above there is an edge from $\frac{p^a}{q^a}$ to $\frac{p^c}{q^c}$

$\therefore$ we get



easy to see $\frac{p}{q} = \frac{p^c + p^a}{q^c + q^a}$

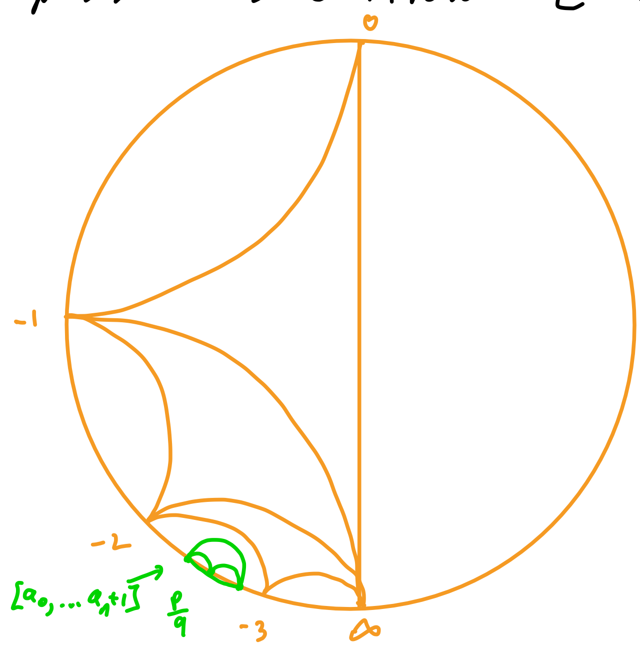
exercise: check $a_n = -2$ case

check case $r > 0$ and r an integer

given $r = \frac{p}{q} = [a_0, \dots, a_n] < -1$ we will be interested in the shortest path in the Farey graph from $\frac{p}{q}$ to -1

note: $[a_0, \dots, a_n+1]$ is the closest point to -1 with an edge to $\frac{p}{q} = [a_0, \dots, a_n]$

since the edge from $[a_0, \dots, a_{n-1}]$ to $[a_0, \dots, a_n+1]$ "shields" $\frac{p}{q}$ from having an edge to a point outside interval $[[a_0, \dots, a_n+1], [a_0, \dots a_n]]$



so if we have a convex torus T with dividing slope $r = [a_0, \dots a_n] < -1$ and we attach a bypass along a ruling curve of slope -1 (or $s \in [-1, 0)$) then the resulting torus will has slope $[a_0, \dots, a_n+1]$

similarly $[a_0, \dots, a_n+2]$ will be closest point to -1 with edge to $[a_0, \dots, a_n+1]$

continuing we see the shortest path from $\frac{p}{q}$ to -1 is given by

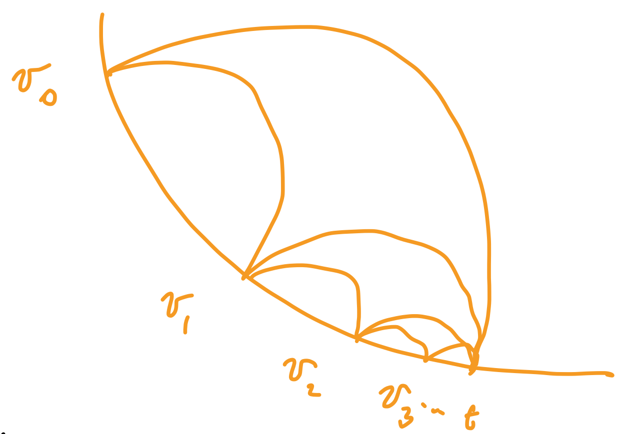
$[a_0, \dots a_n], [a_0, \dots, a_n+1], \dots \; [a_0, a_1+1], \dots$

$[a_0+1], \dots, [-2], [-1]$

note: this gives $|a_n+1| + |a_{n-1}+2| + \dots + |a_1+2|$ edges is the shortest path

given v_0 and $t \in \mathbb{Q}$ with $t < v_0$ and sharing an edge in the Farey graph we define

$$v_k = v_{k-1} \oplus t = v_0 \oplus k t$$



we call the path $v_0 \dots v_k$ a continued fraction block

exercise:

show a path $v_0, \dots, v_k$ is a continued fraction block

$\Leftrightarrow$

there is a change of basis taking it to $-1, -2, \dots, -k-1$

note: $v_{|a_n+1|} = [a_0, \dots a_n]$,

$v_{|a_n+2|} = [a_0, \dots, a_n+1]$, ...,

$\vdots$

$v_0 = [a_0, \dots a_{n-1}, -1] = [a_0, \dots, a_{n-1}+1]$

is a continued fraction block since with

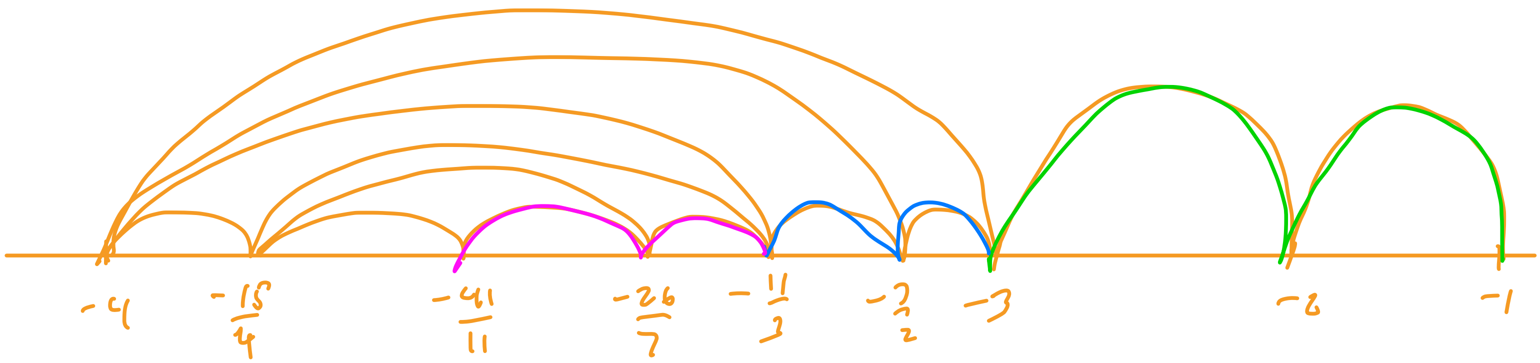
$v_0 = [a_0, \dots, a_n]$ and $t = [a_0, \dots, a_{n-1}]$

example:

$-\frac{41}{11} = [-4,-4,-3]$, $-\frac{26}{7} = [-4,-4,-2]$, $-\frac{11}{3} = [-4,-3]$

$-\frac{11}{3} = [-4,-3]$, $-\frac{7}{2} = [-4,-2]$, $-3 = [-3]$

$-3 = [-3]$, $-2 = [-2]$, $-1 = [-1]$



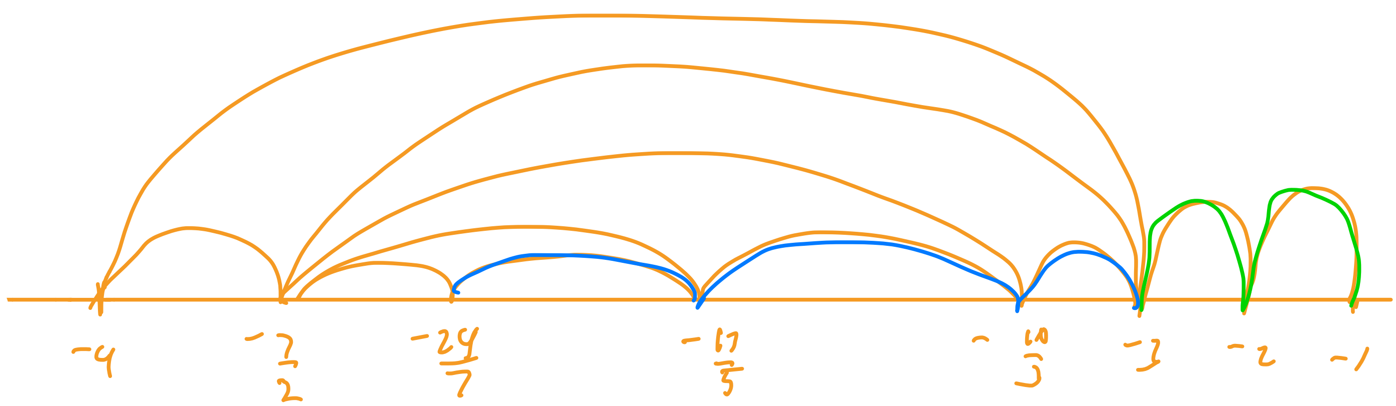
$6 = |-3+1| + |-4+2| + |-4+2|$ edges

3 continued fraction blocks each with 2 edges

example:

$-\frac{24}{7} = [-4,-2,-4]$, $-\frac{17}{5} = [-4,-2,-3]$, $-\frac{10}{3} = [-4,-2,-2]$, $-3 = [-3]$

$-3 = [-3]$, $-2 = [-2]$, $-1 = [-1]$



$5 = |-4+1| + |-2+2| + |-4+2|$ edges

2 continued fraction blocks one with 3 edges and one with 2

E. Tight contact structures on $T^2\times[0,1]$, $S^1\times D^2$, and $L(p,q)$

let $Tight_{min}(T^2\times[0,1]; s_0, s_1)$ be the isotopy classes of minimally twisting tight contact structures on $T^2\times[0,1]$ with $T_i = T^2\times\{i\}$ convex with 2 dividing curves and $\text{slope}(\Gamma_{T_i}) = s_i$.

recall: minimally twisting means any convex torus of in $T^2\times[0,1]$ isotopic to the boundary has dividing slope in $[s_0, s_1]$

Theorem 10:

$$\boxed{\begin{array}{l}\text{if } -\frac{p}{q} = [a_0, \ldots, a_n] < -1, \text{ then} \\ \quad \left|Tight_{min}(T^2\times[0,1]; -p/q, -1)\right| = \left|(a_0+1)\cdots(a_{n-1}+1)a_n\right|\end{array}}$$

note: by changing bases this classifies <u>all</u> minimally twisting contact structures on $T^2\times[0,1]$

let P_{s_0,s_1} be a minimal path in the Farey graph from s_0 clockwise to s_1

we say P_{s_0,s_1} is a <u>decorated path</u> if each edge has been assigned a $+$ or a $-$

we say two decorations on P_{s_0,s_1} differ by <u>shuffling in continued fraction blocks</u> if each continued fraction block contains the same number of $+$ signs (and hence the same number of $-$ signs)

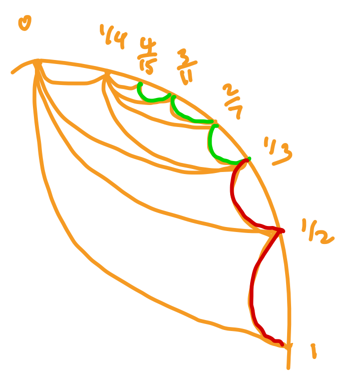
Th^m 10 is equivalent to

Th^m 11:

$Tight_{min}(T^2 \times [0,1]; s_0, s_1)$ is in one-to-one correspondence with decorations on a minimal path in the Farey graph from s_0 clockwise to s_1, upto suffling in continued fration blocks

example:

consider $Tight_{min}(T^2 \times [0,1]; 4/15, 1)$



note: 2 continued fration blocks one of length 2 and other of length 3

so 3 possible sign configurations for first and 4 " " " " second

$\therefore$ we have 12 minimally twisting contact structures

Proof that Th^m 10 and 11 are equivalent:

Th^m 11 $\Rightarrow$ 10: from last section we know that a minimal path from $-\frac{p}{q} = [a_0, \dots, a_n]$ is given by continued fraction blocks

$[a_0, \dots a_n], [a_0, \dots, a_n + 1], \dots [a_0, \dots, a_{n-1}, -1] = [a_0, \dots, a_{n-1} + 1]$

$[a_0, \dots a_{n-1} + 1], [a_0, \dots a_{n-1} + 2], \dots [a_0, \dots, a_{n-2}, -1] = [a_0, \dots, a_{n-2} + 1]$

$\vdots$

$[a_0 + 1], \dots \quad [-1]$

the first continued fraction block has length $|a_n+1|$

and the rest have length $|a_k+2|$

$\therefore$ first has $|a_n|$ sign configurations and rest have $|a_k+1|$

so total number of contact structures is

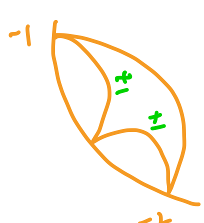
$$|(a_0+1)\cdots(a_{n-1}+1)\,a_n| \checkmark$$

<u>Th$\underline{^m}$ 10 $\Rightarrow$ 11</u>: <u>exercise</u>: show there is a change of basis taking s_1 to -1 and s_0 to a number in $(\infty,-1)$ and this change of basis takes min paths to min paths and continued fraction blocks to continued fraction blocks

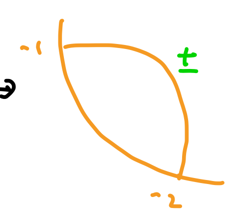
If P is a non-minimal decorated path in the Farey graph then there will be two adjacent edges that can be replaced with a single edge

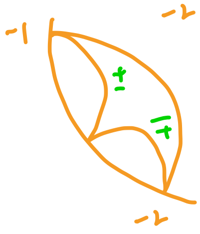
we say the shortening is consistent if the two edges that are replaced have the same sign

in this case the shortened path is also decorated (just give the new edge the sign of the removed edge)



consistent $\rightarrow$





inconsistent

Th$^{\underline{m}}$ 12:

given $\xi \in Tight_{min}(T^2 \times [0,1]; s_0, s_1)$ and
$\xi' \in Tight_{min}(T^2 \times [0,1]; s_1, s_2)$
corresponding to the decorated minimal paths P, P'
if $s_2 \notin [s_0, s_1]$, then the result of gluing ξ and ξ'
together on the torus of slope s_1 will be a tight
minimally twisting contact structure
$\Longleftrightarrow$
$P \cup P'$ can be consistently shortened to a
minimal path

we end the discussion of contact structures on $T^2 \times [0,1]$
with a useful lemma

lemma 13:

given $\xi \in Tight_{min}(T^2 \times [0,1]; s_0, s_1)$
then there is a convex torus isotopic to the
boundary with slope $s \Longleftrightarrow s \in [s_0, s_1]$

we would now like to discuss solid tori
for this we set up some notation
given any slope $s \in \mathbb{Q}^*$
let $S_s = T^2 \times [0,1] / \sim$
where $\sim$ collapses the leaves of the
linear foliation on $T^2 \times \{0\}$ of slope s

exercise: S_s is a solid torus

Hint: • $T^2 \times [0,1] \cong A \times S^1$ where A is an annulus given by a slope s curve on T^2 times $[0,1]$

• collapsing on boundary component of A gives D^2

We say S_s is the solid torus with lower meridian s

Similarly $S^s = T^2 \times [0,1] / \sim$

where $\sim$ collapses the leaves of the linear foliation on $T^2 \times \{1\}$ of slope s

We say S^s is the solid torus with upper meridian s

note: S_∞ is what is normally called a solid torus $S^1 \times D^2$

Thm 14:

$$\boxed{\begin{array}{l} \text{if } -p/q = [a_0, \ldots, a_n] < -1, \text{ then} \\ \qquad |\mathrm{Tight}(S^0; -p/q)| = |(a_0+1)\cdots(a_{n-1}+1)\,a_n| \end{array}}$$

exercise: Show $|\mathrm{Tight}(S_\infty; r)| = |\mathrm{Tight}(S^0; \tfrac{1}{r})|$

hint: consider $f: T^2 \times [0,1] \to T^2 \times [0,1]$

$$(\theta, \phi, t) \longmapsto (\phi, \theta, 1-t)$$

a minimal path P is (upper) mostly decorated if all edges but the last one have a sign and last edge has a 0

it's (lower) mostly decorated if as above but first edge has a 0

Th$^{\underline{m}}$ 14 is equivalent to

Th$^{\underline{m}}$ 15:

let P be a minimal path from r clockwise to s

$\text{Tight}(S^s_j; r)$ is in one-to-one correspondence with (upper) mostly decorations on P upto shuffling signs in continued fraction blocks

$\text{Tight}(S_r; s)$ is the same but use (lower) mostly decorated paths

exercise:

Show Th$^{\underline{m}}$ 14 and 15 are equivalent

(very similar to equivalence of Th$^{\underline{m}}$ 10 and 11

Th$^{\underline{m}}$ 16:

given $\xi \in \text{Tight}(S^m_j; s_0)$ and

$\xi' \in \text{Tight}_{min}(T^2 \times \{0,1\}; s_0, s_1)$

corresponding to the upper mostly decorated minimal path P and decorated path P'

Then the result of gluing ξ, ξ' together along the tori with dividing slope s_0 is tight

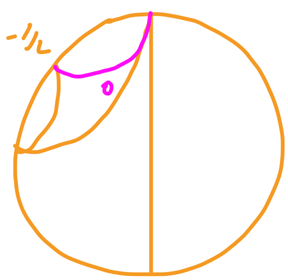
$\Longleftrightarrow$

$s_1 \in [s_0, M]$ and $P \cup P'$ can be consistently shortened to a minimal upper mostly decorated path

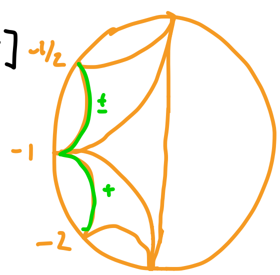
here if one of the edges in the shortening is labled 0 the shortening is consistent and new edge is labled 0

example:

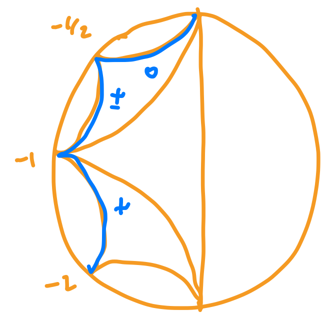
solid torus



$T^2 \times [0,1]$

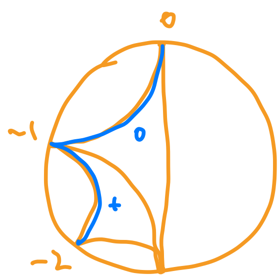


glue $\Rightarrow$



shorten at $-1/2$

consisten for both $\pm$



shortest path so tight

note: If K is a knot in M then a standard nbhd N of K is S_∞ and doing r Dehn surgery is the result of removing $N = S_\infty$ from M and gluing in $S^1 \times D^2$ so the meridian goes to the slope r curve on $\partial(M-N)$.

this is the same as replacing S_∞ with S_r

if U is the unknot in S^3, then $S^3 - \text{nbhd}(U) = S^0$

$\therefore L(p,q) = S^0 \cup S_{-p/q}$

that is $L(p,q) = T^2 \times [0,1] / \sim$

where $\sim$ collapses the leaves of the linear foliation on $T^2\times\{0\}$ of slope $-p/q$ and the leaves of the linear foliation on $T^2\times\{1\}$ of slope 0

Thm 17:

$$\boxed{\begin{array}{l} |\mathrm{Tight}(L(p,q))| = |(a_0+1)\cdots(a_n+1)| \\ \text{where } -p/q = [a_0,\dots,a_n] \end{array}}$$

the main theorems (Thm 10, 14, 17) will follow from

lemma 18:

$$\boxed{\begin{array}{l} |\mathrm{Tight}_{min}(T^2\times[0,1]; -p/q_1, -1)| \le |(a_0+1)\cdots(a_{n-1}+1)a_n| \\ \text{where } -p/q = [a_0,\dots,a_n] < -1 \end{array}}$$

lemma 19:

$$\boxed{\begin{array}{l} \text{if } \frac{p}{q} = [a_0,\dots,a_n] < -1 \text{ and } \frac{p'}{q'} = [a_0,\dots,a_n - 1], \text{ then} \\ |\mathrm{Tight}(L(p',q'))| \le |\mathrm{Tight}(S^0; -p/q)| \le |\mathrm{Tight}_{min}(T^2\times[0,1]; -p/q_1, -1)| \end{array}}$$

Proof of Thms 10, 14, 17:

by constructing Stein fillings of lens spaces in lemma V.2 says

$$|(a_0+1)\cdots(a_{n-1}+1)a_n| \le |\mathrm{Tight}(L(p',q'))|$$

this and lemmas 18, 19 $\Rightarrow$ all contact manifolds under consideration have $|(a_0+1)\cdots(a_{n-1}+1)a_n|$ tight structures up to isotopy! $\blacksquare$

Proof of lemma 19:

given $\xi \in \mathrm{Tight}(S^\circ; -p/q)$

we can Legendrian realize $S^1 \times pt$ in $S^1 \times D^2 = S^\circ$

let N = std nbhd of this Legendrian

so $|\Gamma_{\partial N}| = 2$ and slope $\Gamma_{\partial N}$ is longitudinal

recall this means a curve in $\Gamma_{\partial N}$ and meridian intersect one time, i.e. edge in Farey graph

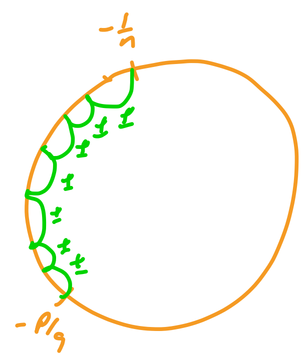
so slope $\Gamma_{\partial N} = \frac{1}{n}$

exercise: by stabilizing we can assume $\frac{1}{n}$ is negative

note: $\overline{S^\circ - N} = T^2 \times [0,1]$ and $\xi|_{T^2 \times [0,1]}$ is minimally twisting

exercise: prove this

so $\xi|_{T^2 \times [0,1]}$ given by



(we prove this and lemma 13 when we prove lemma 18)

so lemma 13 says $\exists$ convex torus $T \subset T^2 \times [0,1]$ isotopic to the boundary with dividing slope -1

let N' = solid torus T bounds

now Kanda's Thm, Thm VIII.5 says $\xi|_{N'}$ is unique

and $\xi|_{\overline{S^\circ - N'}}$ is an elt of $\mathrm{Tight}_{min}(T^2 \times [0,1]; -p/q, -1)$

$\therefore |\mathrm{Tight}(S^\circ; -p/q)| \leq |\mathrm{Tight}_{min}(T^2 \times [0,1]; -p/q, -1)|$

now given $\xi \in Tight(L(p',q'))$ we can think of

$$L(p',q') = S^0 \cup S_{-p'/q'}$$

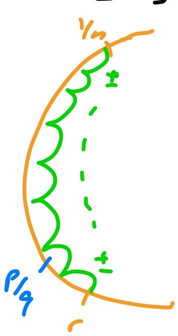
let K = core of $S_{-p'/q'}$ and C = core of S^0

we can Legendrian relize them as L, L', respectively

as above a standard nbhd N' of L' has slope $\frac{1}{n}$

a std nbhd N of L has dividing slope r with an edge to $-p'/q'$ and by stabilizing we can assume r is as close to $-p'/q'$ as we like

now $\overline{L(p,q) - (N \cup N')} = T^2 \times [0,1]$ and $\xi|_{T^2 \times [0,1]}$ minimally twisting so given by



so $\exists$ a convex torus T in $T^2 \times [0,1]$ parallel to the boundary with dividing slope $-p/q$

let S = solid torus T bounds with meridian of slope $-p'/q'$

by Th^m VIII.5 we know $\xi|_S$ is unique and

$\xi|_{\overline{L(p,q) - S}}$ is a tight str on S^0 with dividing slope $-p/q$

$\therefore |Tight(L(p,q))| \leq |Tight(S^0; -p/q)|$

Proof of lemma 18

given $\xi \in \mathrm{Tight}_{min}(T^2\times[0,1]; -p/q, -1)$ where $-\frac{p}{q} = [a_0, \dots, a_n] < -1$

let Γ_i = dividing curves on $T^2 \times \{i\}$

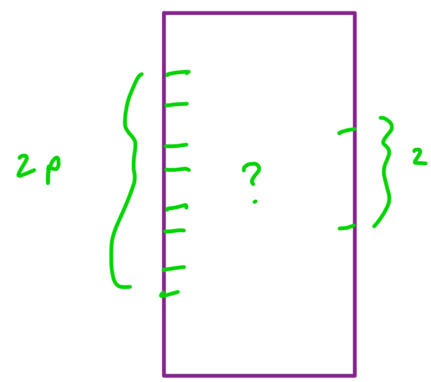
assume $\partial(T^2\times[0,1])$ has ruling curves of slope 0

let $A = S^1 \times [0,1]$ be an annulus s.t. $S^1\times\{i\}$ is a ruling curve on $T^2\times\{i\}$

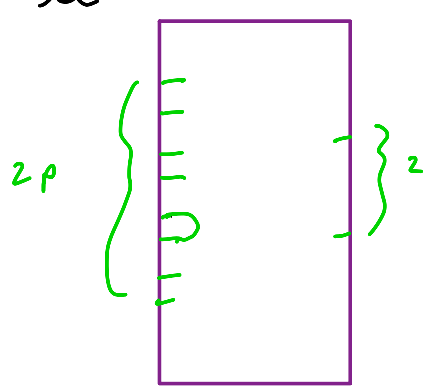
note $tw(S^1\times\{0\}, A) = -\frac{1}{2}\left((S^1\times\{0\})\cap\Gamma_0\right) = -p$

$tw(S^1\times\{1\}, A) = -\frac{1}{2}\left((S^1\times\{1\})\cap\Gamma_1\right) = -1$

so we can make A convex and Γ_A is



so we must see



we can now use Giroux flexiblility to see a bypass on A that we can attach to the front side of $T^2\times\{0\}$

let T' be the result of attaching the bypass

so T' splits $T^2\times[0,1]$ into $(T^2\times[0,1/2])\cup(T^2\times[1/2,1])$

from Corollary 8 and our discussion in the last section we see

T' has 2 dividing curves of slope $-p'/q' = [a_0, \dots, a_n+1]$

so $T^2 \times [0, 1/2]$ is a basic slice with slopes p/q and p'/q'

and $T^2 \times [1/2, 1] \in Tight_{min}(T^2 \times I; -p'/q', -1)$

continuing we can split $(T^2 \times [0,1], \xi)$ into basic slices along

tori of slopes

$[a_0, \dots a_n], [a_0, \dots, a_n+1], \dots [a_0, \dots, a_{n-1}, -1] = [a_0, \dots, a_{n-1}+1]$

$[a_0, \dots a_{n-1}+1], [a_0, \dots a_{n-1}+2], \dots [a_0, \dots, a_{n-2}, -1] = [a_0, \dots, a_{n-2}+1]$

$\vdots$

$[a_0+1], \dots \quad [-1]$

each basic slice has 2 possible contact structures

thus every $\xi \in Tight_{min}(T^2 \times [0,1]; -p/q, -1)$ is obtained by

concatenating basic slices as above

that is, it is given by a decorated minimal path in

the Farey graph from $-p/q$ to -1

so if we see we can shuffle signs in a continued fraction

block then the proof will be complete as discusssed in

the proof of the equivalence of Th$^{\underline{m}}$ 10 and 11

we consider a single continued fraction block and after changing

basis we can assume the slopes of the 2 basic slices are

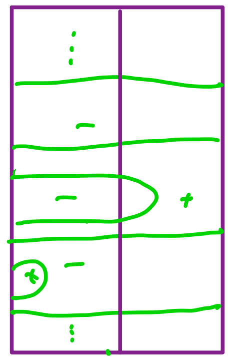
$-n-1, -n, -n+1$

we assume the basic slices have opposite signs

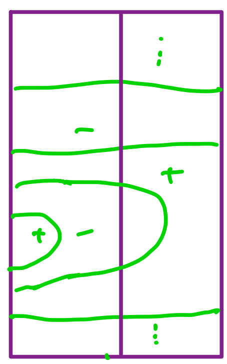
(otherwise there is nothing to prove)

the 2 possibilities for A are

non-nested bypasses

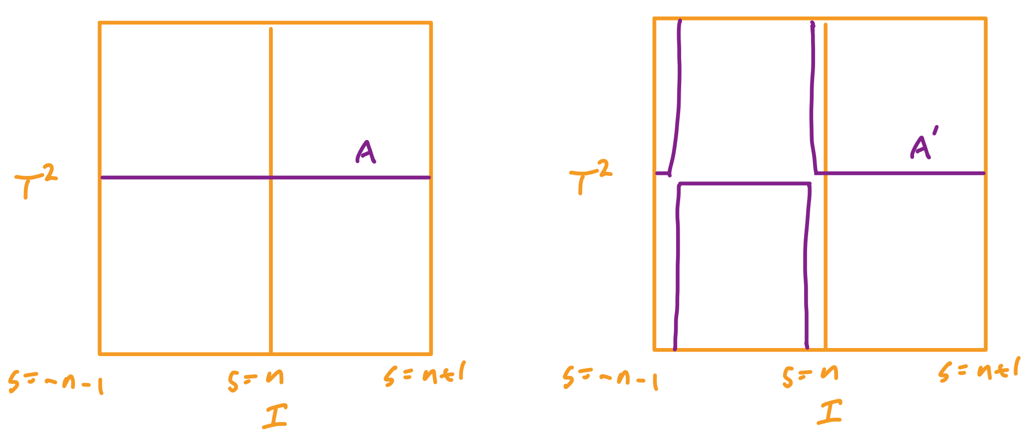


or nested bypasses

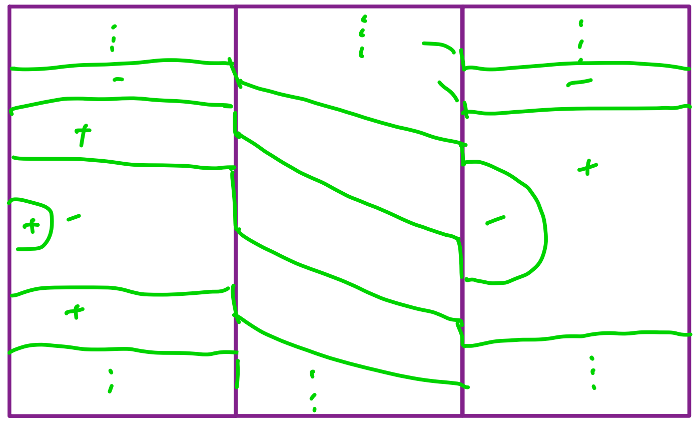


in the first case we can clearly attach the bypasses in any order, i.e. we can shuffle

in the second case we isotop A like in proof of Th^{m} 1



so we "add copies" of torus of slope $-n-1$ and $-n$ to A to get



(we didn't draw $-n-1$ torus as it can't affect nesting)

$\therefore$ we can attach bypasses in any order and hence shuffle signs! $\blacksquare$

lemma 13 was about realizing slopes by $T^2 \subset (T^2 \times [0,1], \xi)$

Proof of lemma 13:

if $(T^2 \times [0,1], \xi)$ is a basic slice with dividing slopes s_0, s_1 then for any $s \in [s_0, s_1]$ there is a convex torus T with slope s (and 2 dividing curves) in $T^2 \times [0,1]$ and isotopic to the boundary

exercise: check this (it follows from the construction of ξ)

since any $\xi \in \mathrm{Tight}_{min}(T^2 \times [0,1]; p/q, -1)$ is a concatenation of basic slices with slopes going from p/q to -1 we are done $\blacksquare$

Proof of Th^{m} 12: ← about gluing contact strs on $T^2\times[0,1]$

suppose we do a consistent shortening

we start by considering a basic slice with slope -2 and -1

by exercise above, in proof of lemma 13, we see $\exists$ a convex torus of slope $-3/2$ that splits $T^2\times[0,1]$ into $T^2\times[0,1/2]$ and $T^2\times[1/2,1]$ and each of these is a basic slice

you can see from the relative Euler class computation in Th^{m} 1 that the sign of the basic slices must be the same

$\therefore$ we see if we can do a consistent shortening we get a tight basic slice and if we can do consistent shortening to get a minimal path with decorations then it must be tight by Th^{m} 11

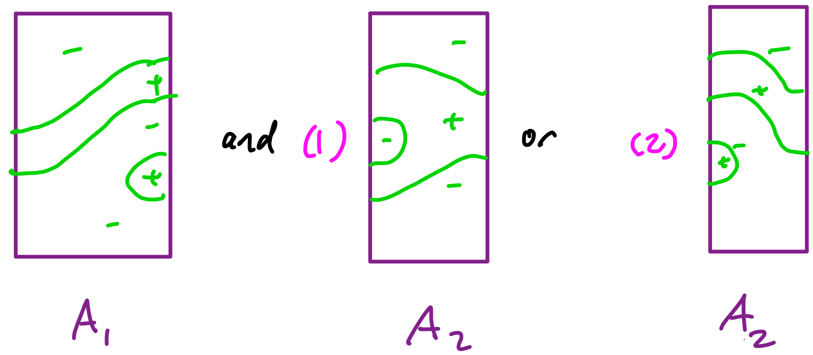
suppose we do an inconsistent shortening, after a change of basis, we can assume the first basic slice has slopes -2 and $-3/2$ while the second has $-3/2$ to -1

so $T^2\times[0,2]=(T^2\times[0,1])\cup(T^2\times[1,2])$ where $\Gamma_i=\Gamma_{T^2\times\{i\}}$ has slope $-2,-3/2,-1$ for $i=0,1,2$, respectively

assume the ruling slope on all the $T^2\times\{i\}$ have slope ∞

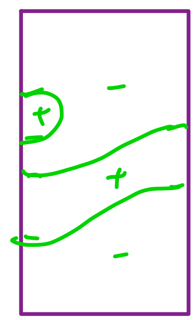
let $A_1=S^1\times[0,1]$ and $A_2=S^1\times[1,2]$ be slope ∞ annuli with boundary ruling curves

we see, after making convex, that Γ_{A_1} and Γ_{A_2} can be

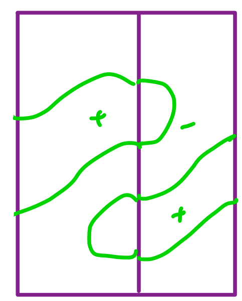


↑ upto changing orientation on A_i can assume this

we can't have A_2 being

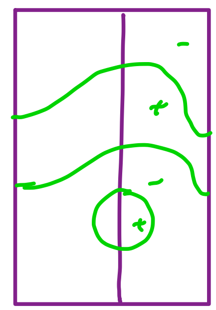


since then $A_1 \cup A_2$ would be



and we could use Giroux flexibility to arrange a curve of slope ∞ that would imply $T^2 \times [0,1]$ was not minimally twisting, but from above we saw a consistant shortening always gives a basic slice (i.e. min. twisting)

in case (2) above we see



so Giroux criterion implies this is overtwisted the other case must be tight from above $\boxplus$

Proof of Th$^{\underline{m}}$ 16: about gluing contact strs on $T^2\times[0,1]$ and $S^1\times D^2$

let $\xi \in \text{Tight}(S^0; -1)$ note: ξ is unique! by Th^{m} VIII.5

from the construction of ξ in Th^{m} VIII.5 we see there is a convex torus T isotopic to ∂S^0 with 2 dividing curves of slope $-1/2$

T splits (S^0, ξ) into the unique element in $\text{Tight}(S^0; -1/2)$ and one of 2 elements in $\text{Tight}_{min}(T^2\times[0,1]; -1, -1/2)$

by reversing orientation on ξ we can assume this basic slice has any sign!

thus shortening a path -1 to $-1/2$ to 0 with a $\pm$ on the first edge and a 0 on second must be tight.

exercise: finish the proof of the theorem
(essentially same as proof above after above observation) $\boxplus$